HISTORY OF MATHEMATICS
MATHEMATICAL TOPIC II
CONSTRUCTIBILITY

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Abstract. We discuss the classical Greek notion of constructibility of geometric objects. The reader is invited to obtain a ruler and compass to perform the exercises and follow the constructions described in the proofs.

1. Construction with Straight-Edge and Compass

The drawings of the ancient Greek geometers were made using two instruments: a straight-edge and a compass.

A *straight-edge* draws lines. With the straightedge, we are permitted to draw a straight line of indefinite length through any two given distinct points. The straight-edge is unmarked; it cannot measure distances.

A *compass* draws circles. With the compass, we are permitted to draw a circle with any given point as the center and passing through any given second point. The compass collapses if it is lifted; we are not *a priori* permitted to use it to measure the distance between given points, and draw a circle around another given point of the same radius.

The straight-edge and the compass have come to be known as *Euclidean tools*, although the quest to construct points using them pre-dates Euclid by two centuries.

2. Construction of Points in a Plane

Let P denote the set of all points in a plane, and let $Q \subset P$.

A line in P is *given by* Q if there exist two points in Q which lie on P.

A circle in P is *given by* Q if the center of the circle is in Q, and there exists a point in Q which lies on the circle.

A point $A \in P$ is *immediately constructible* from Q if one of the following hold:

(a) $A \in Q$;
(b) A is the point of intersection of two lines which are given by Q;
(c) A is a point of intersection of a line and a circle which are given by Q;
(d) A is a point of intersection of two circles which are given by Q.

A point $A \in P$ is *eventually constructible* from Q if there exist a finite sequence of points $A_1, A_2, \dots, A_n$ such that $A = A_n$ and for $j = 1, \dots, n$, A_{j+1} is immediately constructible from $Q \cup \{A_1, \dots, A_j\}$.

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3. Standard Constructions

Let P denote a plane. For $A, B \in P$, define the following:

- AB is the line in P through A and B;
- $\overline{AB}$ is the line segment between A and B;
- $|AB|$ is the distance between A and B;
- $A - B$ is the circle through B with center A.

Also, if $C, D \in P$, then $AB \parallel CD$ represents the statement that line AB is parallel to line CD, and $AB \perp CD$ represents the statement that line AB is perpendicular to line CD.

Let Q be a set of points in the plane. We say that a line segment is constructible from Q if its endpoints are constructible from Q

Proposition 1. *Given points A and B, it is possible to construct the midpoint Z of $\overline{AB}$.*

Construction. We are given A and B.

(a) Let C and D be the points of intersection of circle $A - B$ and circle $B - A$.
(b) Let Z be the intersection of line AB and line CD.

Then Z is the midpoint of $\overline{AB}$. □

Proposition 2. *Given points A and B, it is possible to construct a point Z such that $AB \perp BZ$.*

Construction. We are given A and B.

(a) Let C be the point of intersection of line AB and circle $B - A$ which is not A.
(b) Let Z be one of the points of intersection of circle $A - C$ and circle $C - A$.

Then $AB \perp BZ$. □

Proposition 3. *Given noncolinear points A, B, and C, it is possible to construct a point Z on the line AB such that $AB \perp CZ$.*

Construction. We are given A, B, and C. If $CB \perp AB$, let $Z = C$. Otherwise, construct Z as follows.

(a) Let D be the point of intersection of line AB and circle $C - B$ which is not B.
(b) Let Z be the midpoint of $\overline{BD}$.

Then $AB \perp CZ$. □

Proposition 4. *Given noncolinear points A, B, and C, it is possible to construct a point Z such that $AB \parallel CZ$.*

Construction. We are given A, B, and C.

(a) Let D be the point of intersection of line AB and the line through C which is perpendicular to line AB.
(a) Let Z be the point of intersection of the line through A which is perpendicular to line AB and the line through C which is perpendicular to line CD.

Then $AB \parallel CZ$. □

4. Transference of Distance

Suppose we are given points A, B, and C. A *modern compass* is capable of holding its shape when lifted from the page, so that the distance between A and B can be measured using the modern compass, and then the compass is set down on C to draw a circle with center C and radius $|AB|$. We may call this process *transference of distance*. The Euclidean compass is not *a priori* capable of this feat; however, we can prove that this construction is possible.

Proposition 5. *Given noncolinear points A, B, and C, it is possible to construct a point Z such that polygon $ABCZ$ is a parallelogram.*

Construction. We have points A, B, and C.

- **(a)** Let E be the point of intersection of line BC and circle $B - A$ which lies on the C side of B.
- **(b)** Let F be the midpoint of $\overline{AE}$.
- **(c)** Let G be the point of intersection of line BF and circle $F - B$ which is not B.

Now $|AB| = |BC|$, so $\triangle ABC$ is isosceles. Moreover, F is the midpoint of the base of this isosceles triangle, so $\angle BFE$ is right, whence $\angle EFG$ is right, so $\triangle BFE \cong \triangle GFE$. Similarly, $\triangle AFB \cong \triangle AFG$; thus $|AB| = |BE| = |GE| = |GA|$. Therefore polygon $ABEG$ is a parallelogram, and in particular, line AG is parallel to line BC.

- **(e)** Let Z be the point of intersection of the line through C which is parallel to AB.

Now polygon $ABCZ$ is a parallelogram. □

Proposition 6 (Transference of Distance)**.** *Given points A, B, and C, it is possible to construct a point Z such that $|AB| = |CZ|$.*

Construction. Form parallelogram $ABCZ$. □

5. The Three Greek Problems

As the Greeks investigated what could be accomplished with their Euclidean tools, three interesting unsolved problems arose.

Problem 1 (Duplication of the Cube)**.** Given a cube, construct a cube with double the volume.

Problem 2 (Trisection of an Angle)**.** Given an angle, construct an angle one third as large.

Problem 3 (Quadrature of the Circle)**.** Given a circle, construct a square with the same area.

We now attempt to make the statements of these problems precise, using modern notation.

6. Construction of Squares

A square is constructible if its vertices are constructible.

Quadrature is the process of constructing a square whose area is equal to the area of a given plane region. A plane region with area x is called *quadrable* if it is possible to construct a square with area x. By the Proposition 2, this is equivalent to the the ability to construct a line segment of length $\sqrt{x}$.

The ancient Egyptians estimated areas of certain regions; for example they estimated that the square on 8/9 of the diameter of a circle is its quadrature. The area x of the circle with radius r would then be approximately

$$x \approx \left(\frac{8}{9}(2r)\right)^2 = \frac{256}{81}r^2;$$

this produces $\pi \approx 3.16049$.

The ancient Greeks concentrated on discovering which regions were precisely quadrable, via construction with Euclidean tools.

The third Greek problem asks if a given circle is quadrable.

7. Construction of Angles

Let P denote a plane. For $A, B, C \in P$, define the following:

- $\angle ABC$ is the angle between the line segments $\overline{AB}$ and $\overline{BC}$.

We say that an angle α is constructible from $Q \subset P$ if it is possible to construct points A, B, and C from Q such that $\alpha = \angle ABC$.

To say that an angle α is given; means that we are given points A, B, and C such that $\alpha = \angle ABC$. A *bisector* of this angle is a line BD such that $\angle ABD = \angle DBC$; then necessarily $\angle ABD = \frac{\alpha}{2}$.

Proposition 7. *Given an angle $\angle ABC$, it is possible to construct a point Z such that $\angle ABZ = \angle ZBC = \frac{\angle ABC}{2}$.*

Construction. We are given A, B, and C, with B as the vertex of the angle.

(a) Let D be the point of intersection of BC and $B - C$.

(a) Let Z be the midpoint of $\overline{CD}$.

Then $\angle ABZ = \angle ZBC$. □

Thus every given angle is *bisectable*; the second Greek problem asks if every given angle is *trisectable*.

8. Construction of Points in Space

Let S denote the set of all points in three dimensional space, and let $A, B \in S$. Although the line through A and B is well defined, there are many circles in space whose center is A which pass through B. We do not wish to say that all such circles are constructible.

We say that a plane $P \subset S$ is constructible from a set $Q \subset S$ if there exist three noncolinear points in Q which lie on P. Now circles are constructible from Q if we may construct the plane on which they lie. This gives meaning to the notion of constructibility of a point in space.

A cube is constructible if it is possible to construct its vertices in space.

The first Greek problem asks if, given a cube in space, it is possible to construct a cube in space whose volume is double that of the given cube. This is equivalent to asking if, given a line segment whose length is that of a side of the original cube, it is possible to construct a line segment whose length is that of a cube with double the volume.

9. Construction of Real Numbers

Let P be a plane and let $Q \subset P$. Let $x \in \mathbb{R}$. We say that x is constructible from Q if a line segment whose length is $|x|$ is constructible from Q. Moreover, we say simply that x is a *constructible real number* if x is constructible from $\{A, B\}$ for some $A, B \in P$ with $|AB| = 1$. Since we may consider a point to be a line segment of length 0, we consider 0 to be a constructible number.

Proposition 8. *Let $x, y \in \mathbb{R}$ be constructible. Then $x + y$ is constructible.*

Construction. Since x and y are constructible, it is possible to construct line segments of length $|x|$ and $|y|$. By Proposition 6, it is possible to construct a circle of radius $|y|$ centered at any given point.

(a) Let A and B be points such that $|AB| = |x|$.

Case 1 First assume that x and y have the same sign.

(b) Let Z be the point of intersection of line AB and the circle centered at B of radius y such that B lies on $\overline{AZ}$.

Now $\overline{AZ}$ is a line segment of length $|x| + |y| = |x + y|$.

Case 2 Next assume that x and y have different signs, and without loss of generality assume that $|x| \geq |y|$.

(b) Let Z be the point of intersection of line AB and the circle centered at B of radius y such that Z lies on $\overline{AB}$.

Now $\overline{AZ}$ is a line segment of length $|x| - |y| = |x + y|$. □

Proposition 9. *Let $x \in \mathbb{R}$ be constructible. Then $-x$ is constructible.*

Reason. This follows immediately from the definition. □

Proposition 10. *Let $x, y \in \mathbb{R}$ be constructible. Then xy is constructible.*

Construction. Since 1, x and y are constructible, it is possible to construct line segments of length 1, $|x|$, and $|y|$. Without loss of generality, we may assume that x and y are positive.

- **(a)** Let A and B be points such that $|AB| = x$.
- **(b)** Let C be a point of intersection of the line through A which is perpendicular to line AB and a circle centered at A of radius 1.
- **(c)** Let D be the point of intersection line through AC and the circle centered at C of radius y such that C does not lie on $\overline{AD}$.
- **(d)** Let Z be the intersection of line BC and the line through D which is parallel to AB.

Set $z = |DZ|$; then $\triangle CAB$ is similar to $\triangle CDZ$, so $\frac{1}{x} = \frac{y}{z}$, whence $z = xy$. □

Proposition 11. *Let $x \in \mathbb{R} \setminus \{0\}$ be constructible. Then $\frac{1}{x}$ is constructible.*

Construction. Since 1 and x are constructible, it is possible to construct line segments of length 1 and $|x|$. Without loss of generality, assume that x is positive.

- **(a)** Let A and B be points such that $|AB| = x$.
- **(b)** Let C be the point of intersection of line AB and the circle centered at A of radius 1 such that A is not on $\overline{BC}$.
- **(c)** Let D be a point of intersection of the line through A which is perpendicular to line AB and the circle centered at A of radius 1.
- **(d)** Let Z be the point of intersection of line AD and the line through C which is parallel to line BD.

Set $z = |AZ|$; then $\triangle ZAC$ is similar to $\triangle DAB$, so $\frac{z}{1} = \frac{1}{x}$, that is, $z = \frac{1}{x}$. □

A subset $F \subset \mathbb{R}$ with at least two elements is a *field* if it is closed under the operations of addition, subtraction, multiplication, and division. We have seen that the set of all constructible real numbers is a field. In particular, all rational numbers are constructible. Are there any others?

We show that the set of constructible numbers is closed under square roots; to do this, we need a couple of lemmas. Let's assume the geometric facts that the sum of angles in a triangle is 180°, and that the base angles of an equilateral triangle are equal.

Lemma 1. **(Thales Theorem)** *An angle inscribed in a semicircle is right.*

Proof. Consider a semicircle with center O and diameter $\overline{BC}$, and let A be an arbitrary point on the semicircle; we wish to show that $\angle BAC$ is right. Now $|OA| = |OB| = |OC|$, so $\triangle BOA$ and $\triangle COA$ are isosceles triangles. Let $\alpha = \angle OBA = \angle OAB$ and $\beta = \angle OCA = \angle OAC$; then $\angle BAC = \alpha + \beta$. Adding the angles $\triangle ABC$ we obtain

$$180^\circ = \angle OBA + \angle OCA + \angle BAC = \alpha + \beta + (\alpha + \beta) = 2(\alpha + \beta).$$

Therefore, $\angle BAC = \alpha + \beta = 90^\circ$. □

Lemma 2. *Let $\angle ACB$ be right, and let $D \in \overline{AB}$ such that $AB \perp CD$. Then $\triangle ACB \sim \triangle ADC \sim \triangle CDB$.*

Proof. Two triangles are similar if and only if they have two equal angles. Since $\angle ACB = \angle ADC = \angle CDB = 90^\circ$, $\angle DAC$ is shared by two of the triangles, and $\angle DBC$ is shared by two of the triangles, the result follows. □

Proposition 12. *Let $x \in \mathbb{R}$ be a constructible number. Then $\sqrt{|x|}$ is constructible.*

Construction. Since 1 and x are constructible, it is possible to construct line segments of length 1 and $|x|$. We may assume that x is positive.

- **(a)** Let A and B be points such that $|AB| = x$.
- **(b)** Let C be the point of intersection of line AB and the circle centered at B of radius 1 such that B is on $\overline{AC}$.
- **(c)** Let D be the midpoint of $\overline{AC}$.
- **(d)** Let Z be a point of intersection of the line through B which is perpendicular to line AB and the circle $D - A$.

Let $z = |BZ|$. Now $\angle ZBA = \angle ZBC = 90^\circ$; moreover, $\angle AZC$ is right by Thales theorem. Therefore $\triangle ZBC$ is similar to $\triangle ABZ$. Thus $\frac{z}{x} = \frac{1}{z}$, whence $z^2 = x$, so $z = \sqrt{x}$. □

10. Hippocrates Quadrature of the Lune

Proposition 13. *Any given rectangle is quadrable.*

Construction. Let $BCDE$ form a rectangle. Construct a square as follows:

- **(a)** Let F be the point of intersection of line BE and circle $E - D$ such that $E \in \overline{BF}$.
- **(b)** Let G be the midpoint of $\overline{BF}$.
- **(c)** Let H be the point of intersection of line DE and circle $G - F$ such that $E \in \overline{DH}$.
- **(d)** Let K be the point of intersection of line BE and circle $E - H$ such that $F \in \overline{EK}$.
- **(e)** Let L be the point of intersection of circle $K - E$ with the line through K which is perpendicular to BE such that $EHLK$ forms a polygon.

Then polygon $EHLK$ is a square whose sides have length $a = |HE|$. Let $c = |BG| = |GH|$ and $b = |GE|$. Since $\triangle GEH$ is right, we have $a^2 + b^2 = c^2$. Now

$$\begin{aligned}\text{area}(BCDE) &= |BE| \times |ED| \\ &= |BE| \times |EF| \\ &= (c+b)(c-b) \\ &= c^2 - b^2 = a^2 \\ &= \text{area}(EHLK).\end{aligned}$$

□

Let P denote a plane. For $A, B, C \in P$, define the following:

- $\triangle ABC$ is the triangle whose vertices are A, B, and C.

Proposition 14. *A given triangle is quadrable.*

Construction. Let BCD form a triangle.

- **(a)** Let E be the point of intersection of line BC and the line through D which is perpendicular to BC.
- **(b)** Let F be the midpoint of $\overline{DE}$.
- **(c)** Let G be the point of intersection of the line through F which is parallel to BC and the line through B which is perpendicular to BC.
- **(d)** Let H be the point of intersection of line GF and the line through C which is perpendicular to BC.

Then $BCHG$ form a rectangle whose area is equal to the area of $\triangle BCD$. □

A *lune* is a plane region obtained by taking the complement of one disk with respect to another, where the bounding circles of the disks intersect in two points.

We now produce Hippocrates' lune. The construction uses three ingredients:

(1) the Pythagorean Theorem;
(2) an angle inscribed in a semicircle is right;
(3) the areas of two circles are to each other as the squares on their diameters.

Proposition 15. *Let A and B be points in a plane and let O be the midpoint of $\overline{AB}$. Let C be one of the points of intersection of circle $O-A$ and the line through O which is perpendicular to line AB. Let D be the midpoint of $\overline{AC}$. Let E be the point of intersection of line OD and circle $O-A$ such that $D \in \overline{OE}$. Let F be the point of intersection of line OD and circle $D-A$ such that $F \in \overline{DF}$. Then lune $AECF$ is quadrable.*

Construction. Our goal is to show that area(lune$AECF$) = area($\triangle AOC$). Note that $\angle ACB$ is a right angle, since it is inscribed in a semicircle. Triangles $\triangle AOC$ and $\triangle BOC$ are congruent by SAS; thus $|AC| = |BC|$. Apply the Pythagorean Theorem to get

$$|AB|^2 = |AC|^2 + |BC|^2 = 2|AC|^2.$$

Now

$$\frac{\text{area(semicircle } AFC)}{\text{area(semicircle } ACB)} = \frac{|AC|^2}{|AB|^2} = \frac{|AC|^2}{2|AC|^2} = \frac{1}{2}.$$

A quadrant is half of a semicircle, so clearly

$$\text{area(quadrant } ACO) = \frac{1}{2}\text{area(semicircle } ACB).$$

Thus

$$\text{area(semicircle } AFC) = \text{area(quadrant } ACO).$$

Therefore

$$\begin{aligned} \text{area(lune}AECF) &= \text{area(semicircle } AFC) - \text{area(region } AECD) \\ &= \text{area(quadrant } ACO) - \text{area(region } AECD) \\ &= \text{area}(\triangle ACO). \end{aligned}$$

□

11. Construction of Regular Polygons

A polygon is *regular* if each edge has identical length and the angles at each vertex are equal. For each positive integer n with $n \geq 3$, there is exactly one regular polygon with n edges, up to similarity; is is called a regular n-gon.

Let us first determine the angles in a regular n-gon. It can be inscribed in a circle, and so has a specific center. Divide the n-gon into n isosceles triangles, each with adjacent vertices on the n-gon, with the third vertex being the center. Note that the base angles bisect the angles of the n-gon. Now the sum of the angles of the triangles which come together at the center is $360°$. Thus the base angles add to $180°n - 360°$. There are $2n$ congruent base angles, so each has size

$$\frac{180°n - 360°}{2n} = 90°\Big(1 - \frac{2}{n}\Big).$$

The angles of the n-gon consist of two base angles, so each angle of the n-gon is

$$180°\Big(1 - \frac{2}{n}\Big).$$

We may canonically inscribe a regular polygon with n edges in the unit circle of the cartesian plane; its set of vertices is

$$\{(\cos\alpha, \sin\alpha) \in \mathbb{R}^2 \mid \alpha = \frac{2\pi k}{n} \text{ for } k = 0, 1, \ldots, n-1\}.$$

This is a convenient way for us to view a regular polygon: for example, the length of one side is the distance from $(1, 0)$ to $(\cos\alpha, \sin\alpha)$, where $\alpha = \frac{2\pi}{n}$. By the distance formula,

$$\begin{aligned} \text{length(edge)} &= \sqrt{(\cos\alpha - 1)^2 + (\sin\alpha - 0)^2} \\ &= \sqrt{\cos^2\alpha + \sin^2\alpha + 1 - 2\cos\alpha} \\ &= \sqrt{2 - 2\cos\alpha}. \end{aligned}$$

The ancient Greeks, however, had no coordinate system; they attempted to construct regular polygons using straight-edge and compass.

If a line segment of length r is given, we see that constructibility of a regular n-gon is equivalent to the constructibility of the real number $r\cos\frac{360°}{n}$. We reserve the right to use this existence criterion later, but we begin with actual constructions. All of our constructions proceed from a line segment of length r, and are inscribed in a circle of radius r.

Let O and A be given point with $|OA| = r$. If we can construct a point Z such to $\angle AOZ = \frac{360°}{n}$, then we can complete the construction of the other vertices by intersecting circles centered at a previously constructed vertex of radius $|AZ|$ with circle $O - A$. Thus, it suffices to construct such a point Z.

Proposition 16. *A regular triangle is constructible from* $\{O, A\}$.

Proof. We are given O and A.

(a) Let B be the point of intersection of line OA and circle $O - A$ which is not A.
(b) Let C be the midpoint of $\overline{OB}$.
(c) Let Z be a point of intersection of the line through C which is perpendicular to line OA and circle $O - A$ such that $\angle AOZ \leq 180°$.

Now $\angle AOZ = 120°$. □

Proposition 17. *A square is constructible from* $\{O, A\}$.

Proof. We are given O and A.

(a) Let Z be the point of intersection of the line through O which is perpendicular to line OA and circle $O - A$ such that $\angle AOZ \leq 180°$.

Now $\angle AOZ = 90°$. □

Proposition 18. *If a regular n-gon is constructible, then so is a regular $2n$-gon.*

Construction. We are given O and A.

(a) Let B be a point on the circle $O - A$ such that $\angle AOB = \frac{360°}{n}$.
(b) Let C be the midpoint of $\overline{AB}$.
(c) Let Z be the intersection of line OC and circle $O - A$ such that Z lies on $\overline{AB}$.

Now $\angle AOZ = \frac{360°}{2n}$. □

Thus we may construct regular triangles, quadrilaterals, and hexagons. We would like to know if a regular pentagon is constructible. Investigating this brings us to the world of the golden ratio.

12. Exercises

For each construction, provide a drawing produced with an actual straight-edge and compass, together with a list of steps sufficient to reproduce the drawing (as in the propositions of the text). If you apply the propositions to construct a midpoint or perpendicular, use a marked ruler or protractor to obtain a more accurate picture.

Exercise 1. For each subset Q of a plane P, find all points that are immediately constructible from Q.

(a) Q consists of two points
(b) Q consists of the vertices of an equilateral triangle
(c) Q consists of the vertices of an isosceles triangle

Exercise 2. Reproduce the drawings which correspond to the construction instructions for the following propositions.

(a) Proposition 1 (midpoints)
(b) Proposition 3 (perpendiculars)
(c) Proposition 5 (transference of distance)
(d) Proposition 10 (products of constructible lengths)
(e) Proposition 11 (quotients of constructible lengths)
(f) Proposition 12 (square roots of constructible lengths)
(g) Proposition 13 (quadrature of a rectangle)
(h) Proposition 14 (quadrature of a triangle)
(i) Proposition 15 (quadrature of a lune)

Exercise 3. Given circle $A - B$, construct an equilateral triangle inscribed in the circle with one vertex at B.

Exercise 4. Given circle $A - B$, construct a regular hexagon inscribed in the circle with one vertex at B.

Exercise 5. Given three noncollinear points, construct the center of the unique circle which contains the three points.

Exercise 6. Given two points, construct a line segment of length $\sqrt{3}$.

Exercise 7. Given two points, construct a line segment of length $\sqrt{2}$.

Exercise 8. Given two points, construct an angle of $45°$.

Exercise 9. Given two points, construct an angle of $75°$.

Exercise 10. Given a circle, construct a concentric circle with quadruple the area.

Exercise 11. Given a circle, construct a concentric circle with triple the area.

Exercise 12. Given a circle, construct a concentric circle with double the area.

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